Find the 13^{th} term and the sum of the first 9 terms of the geometric sequence $48, -24, 12, -6, 3, -\frac{3}{2}, \ldots$.

Solution

We have $a = 48$ and $r = -\frac{1}{2}$. Using part (a) of Theorem 3.2, we find that the 13^{th} term is $48(-\frac{1}{2})^{12} = \frac{3}{256}$. Using (3.4d), the sum of the first 9 terms is $48\left[\frac{1-(-\frac{1}{2})^9}{1-(-\frac{1}{2})}\right] = 48\left[\frac{1+\frac{1}{512}}{\frac{3}{2}}\right] = \frac{513}{16}$. □

3.2 BASIC RESULTS

John borrows 1500 from a finance company and wishes to pay it back with equal annual payments at the end of each of the next ten years. If $i = .17$, what should his annual payment be?

Jacinta buys a house and takes out a 50,000 mortgage. If the mortgage rate is 13% convertible semiannually, what should her monthly payment be to pay off the mortgage in 20 years?

Eileen deposits 2000 in a bank account every year for 11 years. If $i = .06$, how much has she accumulated at the time of the last deposit?

All of these questions have one thing in common: they involve a series of payments made at regular intervals. Such a series of payments is called an *annuity*. In the three cases above, the payments are of equal amount, and that will be the case with all annuities studied in this section. Later, however, we will study more general annuities.

Annuities turn up in many different types of financial transactions. From the point of view of practical applications, a complete understanding of annuities is an absolute must!

We shall start by considering an annuity under which payments of 1 are made at the end of each period for n periods. Sometimes a period will be one year, as with John's loan above, but other periods are certainly possible. It will be assumed throughout that, as with John's loan, the interest period and the payment period are equal. When this is not the case, as with Jacinta's mortgage, for example, we will first find the equivalent rate of interest per payment period and then proceed with our solution.

Level payments of an amount other than 1 can be handled by multiplying by the amount of the payment, as we shall see in the examples.

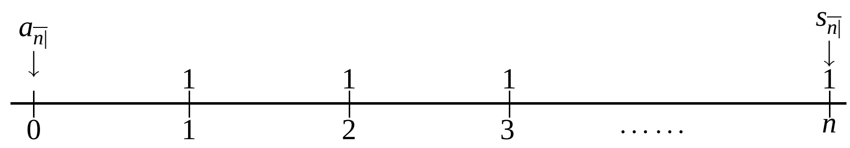

FIGURE 3.1

A time diagram showing n payments of 1 is given in Figure 3.1. The present value of this annuity at time 0 is denoted by $a_{\overline{n}|}$. The accumulated value of this annuity at time n is denoted by $s_{\overline{n}|}$.

We shall now derive a formula for $a_{\overline{n}|}$. Taking the value at time 0 of each of the payments in turn, we obtain

$$a_{\overline{n}|} = v + v^2 + v^3 + \cdots + v^n. \tag{3.5}$$

This is the sum of n terms of a geometric sequence with $a = v$ and $r = v$. Using Formula (3.4d) developed in Section 3.1, we obtain

$$\begin{aligned} a_{\overline{n}|} &= \frac{v(1-v^n)}{1-v} \\ &= \frac{1-v^n}{\frac{1}{v}-1} \\ &= \frac{1-v^n}{1+i-1} \\ &= \frac{1-v^n}{i}. \end{aligned} \tag{3.6}$$

Formula (3.6) is crucial, and will be used frequently throughout the rest of the text.

It is easy now to get a formula for $s_{\overline{n}|}$. Since $s_{\overline{n}|}$ is the value of the same annuity n years after $a_{\overline{n}|}$ has been calculated, it follows that

$$\begin{aligned} s_{\overline{n}|} &= a_{\overline{n}|}(1+i)^n \\ &= \left[\frac{1-v^n}{i}\right](1+i)^n \\ &= \frac{(1+i)^n - v^n(1+i)^n}{i} \\ &= \frac{(1+i)^n - 1}{i}. \end{aligned} \tag{3.7}$$

Let us immediately proceed to some practical examples.

Example 3.3

Find John's payment in the problem stated in the first paragraph of this section.

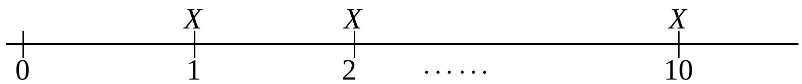

FIGURE 3.2

Solution

Let the payment be X. Since the present value of 10 payments of 1 is $a_{\overline{10|}}$, the present value of 10 payments of X will be $X \cdot a_{\overline{10|}}$. Thus we have

$$1500 = X \cdot a_{\overline{10|}}. \text{ Then } X = \frac{1500}{a_{\overline{10|}}} = \frac{1500}{\frac{1-v^{10}}{i}} = \frac{1500(.17)}{1-(\frac{1}{1.17})^{10}} = 321.98.$$

□

Example 3.4

Find the accumulated value in Eileen's bank account in the problem stated in the third paragraph of this section.

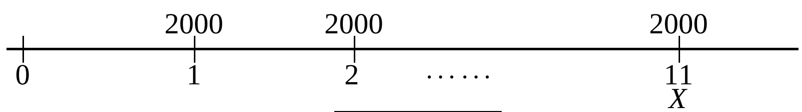

FIGURE 3.3

Solution

Since each deposit is 2000, the accumulated value will be given directly by $X = 2000s_{\overline{11|}} = 2000\left[\frac{(1.06)^{11}-1}{.06}\right] = 29{,}943.29.$ □

Example 3.5

Find Jacinta's mortgage payment in the problem stated in the second paragraph of this section.

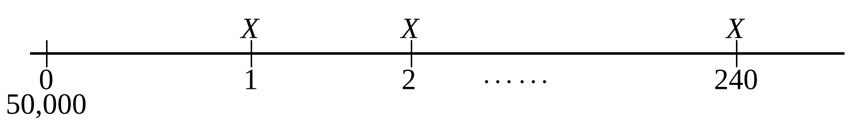

FIGURE 3.4

Solution

As mentioned earlier, we first have to find the effective monthly rate of interest equivalent to 13% convertible semiannually. This is because our formulae for $a_{\overline{n|}}$ and $s_{\overline{n|}}$ are based on the assumption that the interest period and payment period are the same. Letting this monthly rate be j, we have $1+j = \left[1+\frac{.13}{2}\right]^{1/6}$. Now we let the mortgage payment be X. Note that there are 240 monthly payments in the 20-year term of the mortgage, so we have $X \cdot a_{\overline{240|}} = 50{,}000$ and $X = \dfrac{50{,}000j}{1-v^{240}} = 573.77$. □

Example 3.6

Elroy takes out a loan of $5000 to buy a car. No payments are due for the first 8 months, but beginning with the end of the 9*th* month, he must make 60 equal monthly payments. If $i = .18$, find (a) the amount of each payment; (b) the amount of each payment if there is no payment-free period, (i.e., if the first payment is due in one month and the remaining 59 are made on a monthly basis thereafter).

Solution

(a) We first note that a monthly rate of interest j is required. Since $(1+j)^{12} = 1.18$, we obtain $j = (1.18)^{1/12} - 1$. Let the amount of each payment be X.

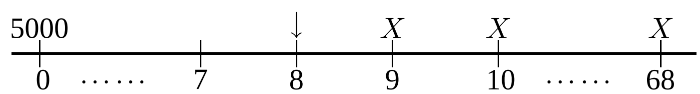

FIGURE 3.5

We now observe that this does not fit into the standard annuity pattern, since $X \cdot a_{\overline{60|}}$ will give us the value of the payments at month 8, one month before the first payment. The value of the loan at time 8 is $5{,}000(1+j)^8$, since it will accrue interest for eight months, even though no payments are required. Thus we have the equation of value $X \cdot a_{\overline{60|}} = 5000(1+j)^8$, so that $X = \dfrac{5000(1+j)^8}{a_{\overline{60|}}}$.

Evaluating $a_{\overline{60|}}$, $X = \dfrac{5000(1+j)^8 \cdot j}{1-v^{60}} = 137.76$.

(b) In this case, we just have $X \cdot a_{\overline{60|}} = 5000$, which we solve for $X = \dfrac{5000j}{1-(1+j)^{-60}} = 123.37$. This shows that, over the next 5 years, the total amount of extra money paid for postponing the first payment for 8 months will be 863.40! □

The previous example beautifully demonstrates the power of our calculators. In pre-calculator days the evaluation of the term $a_{\overline{60|}} = \dfrac{1-(1+j)^{-60}}{j}$ would have caused serious difficulties. Even interest tables would not have helped, because the interest rate j is not one for which tables were constructed. All we do now, however, is press a few buttons (in the right order) and the answer appears!

Example 3.7

(a) Prove the identity $1 = ia_{\overline{n|}} + v^n$.

(b) Give a verbal interpretation of this identity.

Solution

(a) $a_{\overline{n|}} = \dfrac{1-v^n}{i}$, so $ia_{\overline{n|}} = 1 - v^n$, and $1 = ia_{\overline{n|}} + v^n$, as required.

(b) The term $ia_{\overline{n|}}$ can be thought of as the present value of an annuity with level payment i at the end of each year for n years (see Figure 3.6). The term v^n is the present value of 1 at year n.

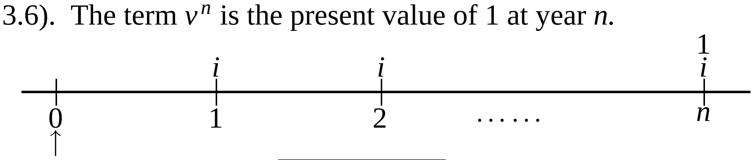

FIGURE 3.6

Imagine investing 1 at time 0. At the end of the first year, the interest is separated off from the original investment, and the amount of the investment is back to 1. This procedure continues for n years, leaving 1 at the end of n years and the annuity of i which was removed each year. The present value of these terms is v^n and $ia_{\overline{n|}}$, respectively. □

There are two other symbols in common usage with annuities, namely $\ddot{a}_{\overline{n|}}$ and $\ddot{s}_{\overline{n|}}$.

$\ddot{a}_{\overline{n}|}$ is the present value of the annuity described earlier at the time of the first payment, and $\ddot{s}_{\overline{n}|}$ is the accumulated value one year after the last payment has been made. Our four functions, $a_{\overline{n}|}$, $\ddot{a}_{\overline{n}|}$, $s_{\overline{n}|}$ and $\ddot{s}_{\overline{n}|}$ are illustrated in Figure 3.7.

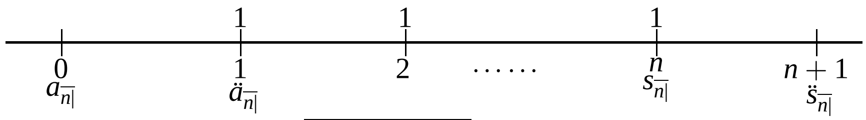

FIGURE 3.7

Mathematically, there is nothing very exciting going on here. We can see immediately from Figure 3.7 that $\ddot{a}_{\overline{n}|} = a_{\overline{n}|}(1+i)$, and that $\ddot{s}_{\overline{n}|} = s_{\overline{n}|}(1+i)$. These relationships lead to formulae for $\ddot{a}_{\overline{n}|}$ and $\ddot{s}_{\overline{n}|}$ that are analogous to Formulae (3.6) and (3.7) for $a_{\overline{n}|}$ and $s_{\overline{n}|}$, respectively. We have

$$\begin{aligned}
\ddot{a}_{\overline{n}|} &= a_{\overline{n}|}(1+i) \\
&= \frac{1-v^n}{i}(1+i) \\
&= \frac{1-v^n}{\frac{i}{1+i}} \\
&= \frac{1-v^n}{d}, \qquad (3.8)
\end{aligned}$$

where d is the effective rate of discount defined in Chapter 1. Similarly, the reader should show that

$$\ddot{s}_{\overline{n}|} = \frac{(1+i)^n - 1}{d}. \qquad (3.9)$$

Observe that $\ddot{a}_{\overline{n}|}$ can also be described as the present value of payments of 1 made at the beginning of each period for n periods, and $\ddot{s}_{\overline{n}|}$ can be described as the accumulated value of the same payments at the end of the last period. Since the payments are at the beginnings of the periods, it follows that $\ddot{s}_{\overline{n}|}$ is their accumulated value a full period after the last payment.

There are many identities relating the four quantities we have introduced. In addition to the ones mentioned earlier, we note that

$$\ddot{s}_{\overline{n|}} = \ddot{a}_{\overline{n|}}(1+i)^n \tag{3.10}$$

and

$$1 = d\ddot{a}_{\overline{n|}} + v^n, \tag{3.11}$$

both of which have nice verbal interpretations. Other relationships will be presented as exercises.

Let us do an example. Imagine that Henry takes out a loan of 1000 and repays it with 10 equal yearly payments, the first one due at the time of the loan. In this case, if X is the amount of each payment, an appropriate equation of value would be

$$X \cdot \ddot{a}_{\overline{10|}} = 1000. \tag{3.12a}$$

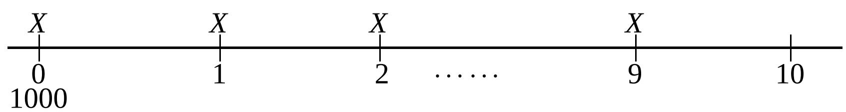

FIGURE 3.8

We use $\ddot{a}_{\overline{10|}}$ here because the annuity symbol $a_{\overline{10|}}$ assumes we are taking the present value *one year before* the first payment, and that is not the case with Henry's loan. An equally good equation of value would be

$$X \cdot a_{\overline{10|}} = 1000v, \tag{3.12b}$$

because $1000v$ would be the value of the loan one year earlier, which allows us to use our first annuity symbol. Once the rate of interest i is known, we can find the amount of Henry's loan payment.

Let us observe as well that a third acceptable equation of value for Henry's problem would be

$$X(1 + a_{\overline{9|}}) = 1000. \tag{3.12c}$$

Example 3.8

Using all three equations of value, find Henry's loan payment if $i = .16$.

Solution

(a) First we consider Equation (3.12a), which is $X \cdot \ddot{a}_{\overline{10|}} = 1000$. Then $X = \frac{1000}{\ddot{a}_{\overline{10|}}} = \frac{1000d}{1-v^{10}}$. To use this approach we need to find $d = \frac{i}{1+i} = \frac{.16}{1.16}$. This leads to the answer $X = 178.36$.

(b) Equation (3.12b) states that $X \cdot a_{\overline{10|}} = 1000v$. Then we have
$$X = \frac{1000\left(\frac{1}{1.16}\right)(.16)}{1-\left(\frac{1}{1.16}\right)^{10}} = 178.36.$$

(c) Finally, consider Equation (3.12c), which gives us the equation $X = \frac{1000}{1+a_{\overline{9|}}} = \frac{1000}{1+\frac{1-v^9}{i}} = 178.36.$ □

The moral of Example 3.8 is that there is more than one way to work out this kind of problem. However, we reiterate the importance of keeping as many decimal places as possible during your calculations.

In many textbooks, the term *annuity-immediate* is used for the case where payments are made at the end of the period, and *annuity-due* is used when payments are made at the beginning of the period. As we have just illustrated, however, the same techniques can be used in both cases.

The following example illustrates that there are many possible ways of analyzing annuities.

Example 3.9

Consider an annuity which pays 1 at the beginning of each year for $m+n$ years. Explain verbally why each of the following expressions gives the *current value* of this annuity at the end of year m. (See Figure 3.9 on the following page.)

(a) $a_{\overline{m+n|}}(1+i)^{m+1}$

(b) $\ddot{a}_{\overline{m+n|}}(1+i)^m$

(c) $s_{\overline{m+n|}}\, v^{n-1}$

(d) $\ddot{s}_{\overline{m+n|}}\, v^n$

(e) $s_{\overline{m+1|}} + a_{\overline{n-1|}}$

(f) $\ddot{s}_{\overline{m|}} + \ddot{a}_{\overline{n|}}$

(g) $1 + \ddot{s}_{\overline{m|}} + a_{\overline{n-1|}}$

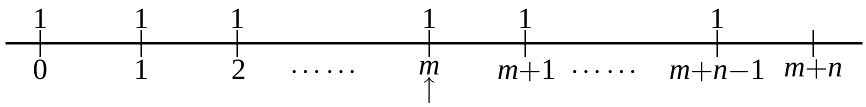

FIGURE 3.9

Solution

In the time diagram for this annuity in Figure 3.9, we have denoted by ↑ the point at which we want the value of the annuity.

(a) $a_{\overline{m+n}|}$ is the value at year -1. To get to ↑, we must move $m+1$ years into the future, so we have $a_{\overline{m+n}|}(1+i)^{m+1}$.

(b) $\ddot{a}_{\overline{m+n}|}$ is the value at year 0. So $\ddot{a}_{\overline{m+n}|}(1+i)^{m}$ moves us m years into the future.

(c) $s_{\overline{m+n}|}$ is the value at year $m+n-1$. Hence we move back $n-1$ years.

(d) $\ddot{s}_{\overline{m+n}|}$ is the value at year $m+n$, so we move back n years.

(e) $s_{\overline{m+1}|}$ is the value of the first $m+1$ payments at time m, and $a_{\overline{n-1}|}$ is the value of the last $n-1$ payments at time m.

(f) $\ddot{s}_{\overline{m}|}$ is the value of the first m payments at time m, and $\ddot{a}_{\overline{n}|}$ is the value of the last n payments at the same time.

(g) Here the single payment of 1 at time m is separated off from the $\ddot{a}_{\overline{n}|}$ in part (f), leaving $a_{\overline{n-1}|}$. □

The above example should illustrate how careful we must be when working with these functions, but also that we have considerable flexibility in using them to express an annuity value at some point of time.

3.3 PERPETUITIES

A *perpetuity* is an annuity whose payments continue forever. The time diagram is shown in Figure 3.10 below.

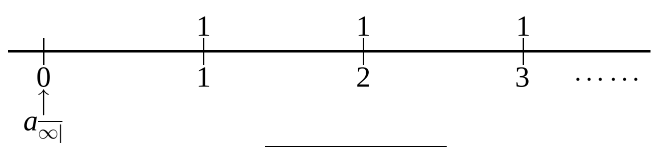

FIGURE 3.10

The value of this annuity one year before the first payment is $a_{\overline{\infty|}}$. We have

$$\begin{aligned} a_{\overline{\infty|}} &= \lim_{n\to\infty} a_{\overline{n|}} \\ &= \lim_{n\to\infty} \frac{1-v^n}{i} \\ &= \frac{1}{i}, \qquad (3.13) \end{aligned}$$

since $\lim_{n\to\infty} v^n = 0$, as long as $i > 0$.

We can see verbally why Formula (3.13) should be true: if a principal of $\frac{1}{i}$ is invested at rate i, then the interest $\left(\frac{1}{i}\right) i = 1$ can be removed at the end of each year, leaving the original principal intact forever.

As in Section 3.2, the symbol $\ddot{a}_{\overline{\infty|}}$ represents the value of a perpetuity at the time of the first payment. The following identities are left as exercises for the reader:

$$\ddot{a}_{\overline{\infty|}} = a_{\overline{\infty|}}(1+i), \qquad (3.14)$$

$$\ddot{a}_{\overline{\infty|}} = 1 + a_{\overline{\infty|}}, \qquad (3.15)$$

and

$$\ddot{a}_{\overline{\infty|}} = \frac{1}{d}. \qquad (3.16)$$

3.4 UNKNOWN TIME AND UNKNOWN RATE OF INTEREST

We will consider here several examples involving annuities where the length of time or the rate of interest involved is the unknown.

Example 3.10

A fund of 5000 is used to award scholarships of amount 500, one per year, at the end of each year for as long as possible. If $i = .09$, find the number of scholarships which can be awarded, and the amount left in the fund one year after the last scholarship has been awarded.

(a) The arithmetic sequence 2, 7, 12, 17, ...
(b) The arithmetic sequence with $a = 71$ and $d = -3$.
(c) The arithmetic sequence whose 5^{th} term is 19 and whose 9^{th} term is 47.
(d) The geometric sequence 5, 15, 45, ...
(e) The geometric sequence $3, -\frac{3}{4}, \frac{3}{16}, -\frac{3}{64}, \ldots$.
(f) The geometric sequence whose 5^{th} term is $\frac{2}{9}$ and whose 8^{th} term is $\frac{3}{32}$.

3-2. Prove Theorems 3.1(a), 3.1(b), 3.2(a) and 3.2(b) using mathematical induction.

3.2 Basic Results

3-3. Henrietta borrows 6500 in order to buy furniture. She wishes to pay the loan back by means of 12 annual payments, the first to be made one year after the loan is taken out. If $i = .13$, find the amount of each payment.

3-4. Answer Question 3 if the loan is to be paid back with 144 monthly payments, the first one due one month after the loan is taken out.

3-5. Alphonse deposits 450 in a bank account at the beginning of each year, starting in 1977 and continuing for 20 years. If $i = .08$, find the amount in his account at the end of 1996.

3-6. An annuity pays 1000 a year for 8 years. If $i = .08$, find each of the following:
(a) The value of the annuity one year before the first payment.
(b) The value of the annuity one year after the last payment.
(c) The value of the annuity at the time of the fifth payment.
(d) The number of years the annuity would have to run in order that its current present value be doubled.
(e) The number of years the annuity would have to run in order that its current present value be tripled.

3-7. Prove each of the following identities:
(a) $a_{\overline{m+n|}} = a_{\overline{m|}} + v^m a_{\overline{n|}}$
(b) $a_{\overline{m-n|}} = a_{\overline{m|}} - v^m s_{\overline{n|}}$
(c) $s_{\overline{m+n|}} = s_{\overline{m|}} + (1+i)^m s_{\overline{n|}}$
(d) $s_{\overline{m-n|}} = s_{\overline{m|}} - (1+i)^m a_{\overline{n|}}$

3-8. Give verbal interpretations for each of the identities in Question 7.

3-9. Prove that $\frac{1}{a_{\overline{n}|}} = \frac{1}{s_{\overline{n}|}} + i$.

3-10. Prove each of the following identities:
(a) $\ddot{a}_{\overline{n}|} = 1 + a_{\overline{n-1}|}$
(b) $\ddot{s}_{\overline{n}|} = s_{\overline{n+1}|} - 1$

3-11. Give verbal interpretations for the identities in Question 10.

3-12. Rank n, $a_{\overline{n}|}$ and $s_{\overline{n}|}$ in increasing order of magnitude. Under what conditions will equality hold for all n?

3-13. Harriet wishes to accumulate 85,000 in a fund at the end of 25 years. If she deposits 1000 in the fund at the end of each of the first 10 years, and $1000 + x$ at the end of each of the last 15 years, find x if the fund earns 7% effective.

3-14. Show that $\frac{s_{\overline{2n}|}}{s_{\overline{n}|}} + \frac{s_{\overline{n}|}}{s_{\overline{2n}|}} - \frac{s_{\overline{3n}|}}{s_{\overline{2n}|}} = 1$.

3-15. Prove each of the following identities:
(a) $\ddot{a}_{\overline{n}|} = a_{\overline{n}|} + 1 - v^n$
(b) $\ddot{s}_{\overline{n}|} = s_{\overline{n}|} - 1 + (1+i)^n$

3-16. Give verbal interpretations for the identities in Question 15.

3-17. Show that $\sum_{t=a}^{b}(\ddot{s}_{\overline{t}|} - s_{\overline{t}|}) = s_{\overline{b+1}|} - s_{\overline{a}|} - (b+1-a)$.

3-18. An annuity runs for 25 years as follows: at the end of each of the first ten years 500 is paid, and then at the end of each of the last 15 years 300 is paid. If $i = .08$, find the value of this annuity three years before the first payment.

3-19. Edward buys a new house and takes out a mortgage of 60,000. To pay off the mortgage, he will make monthly payments with the first payment due in one month. Given $i^{(2)} = .12$, find the amount of his payment if (a) the payments will continue for the next 25 years; (b) the payments will continue for the next 20 years; (c) the payments will continue for the next 10 years.

3-20. Rework Question 19 if the nominal rate of interest convertible semiannually is 16% instead of 12%.

3-21. A man wishes to accumulate a small pension by depositing 2500 at the beginning of each year for 25 years. Starting at the end of the year in which the final deposit is made, he will make 20 annual withdrawals. Find the amount of each withdrawal, if $i = .07$ during the first 25 years and $i = .11$ thereafter.

3-22. A series of $n+1$ payments are made as follows: 1 at the end of the first year, 2 at the end of each of the next $n-1$ years, and 1 at the end of year $n+1$. Show that the value of these payments at $t = 0$ is $a_{\overline{n}|} \cdot \ddot{a}_{\overline{2}|}$.

3-23. Give a verbal explanation of why the formula in Question 22 is correct.

3-24. An annuity consists of n payments of 1, the first to be made at the end of 7 years and the other payments to be made at three year intervals thereafter. Show that the present value of the annuity is $\dfrac{a_{\overline{3n+7}|} - a_{\overline{7}|}}{a_{\overline{3}|}}$.

3-25. Albert Glover, star third baseman with the Blue Jays, is given a choice of contracts:

(a) 3,200,000 per year for the next five years, payable at the end of each year.

(b) 3,000,000 per year for the next five years, payable at the beginning of each year.

(c) 1,800,000 per year for the next ten years, payable at the end of each year.

If $i = .04$, find the value of each of these contracts at the beginning of the first year. Repeat for $i = .06$.

3-26. Find the range of interest rates for which each of the contracts in Question 25 has a higher present value then the other two.

3-27. Consider an annuity where $\frac{n}{k}$ payments of 1 are made, the first occurring k years from now with the payments continuing at k-year intervals thereafter, until a period of n years has passed. Prove that the present value of these payments is equal to $\frac{a_{\overline{n}|}}{s_{\overline{k}|}}$.

3-28. Show that the accumulated value of the annuity in Question 27 immediately after the last payment is $\frac{s_{\overline{n}|}}{s_{\overline{k}|}}$.

3-29. Give verbal interpretations for the formulae in Question 27 and Question 28.

3-30. Prove that the present value of an annuity which pays $\frac{1}{m}$ at the end of each m^{th} of a year for the next n years is equal to $\frac{1-v^n}{i^{(m)}}$. This present value is denoted by $a^{(m)}_{\overline{n}|}$.

3-31. Prove that the accumulated value of the annuity in Question 30 at the time of the last payment is $\frac{(1+i)^n-1}{i^{(m)}}$. This accumulated value is denoted by $s^{(m)}_{\overline{n}|}$.

3-32. Derive an expression for the present value of an annuity under which payments are 2, 1, 2, 1, ... at the end of every year for the next 25 years.

3-33. If $a_{\overline{n}|} = x$ and $a_{\overline{2n}|} = y$, express d as a function of x and y.

3-34. A loan of 25,000 is to be repaid by annual payments at the end of each year for the next 20 years. During the first 5 years the payments are k per year; during the second 5 years the payments are $2k$ per year; during the third 5 years, $3k$ per year; and during the fourth 5 years, $4k$ per year. If $i = .12$, find k.

3-35. Given $a_{\overline{n}|} = 12$ and $a_{\overline{2n}|} = 21$, find $a_{\overline{4n}|}$.

3-36. Given $\ddot{a}_{\overline{t|}} = 9.370$ and $\ddot{a}_{\overline{t+1|}} = 9.499$, find the effective rate of interest.

3-37. An injured worker submits a Workers Compensation claim. It is decided that she is entitled to annual medical payments of 20,000 for the next 10 years and equal annual indemnity payments for the next 20 years. The medical payments will begin immediately, and the indemnity payments will begin in one year's time. The insurance company has established a fund of 680,000 to support these payments. Find the amount of each annual indemnity payment assuming $i = .07$.

3.3 Perpetuities

3-38. Prove identities (3.14), (3.15) and (3.16).

3-39. Given $i = .15$, find the present value of an annuity of 100 per year continuing forever if (a) the first payment is due in one year; (b) the first payment is due immediately; (c) the first payment is due in 5 years.

3-40. A perpetuity of 500 per year, with the first payment due one year hence, is worth 2500. Find i.

3-41. Deposits of 1000 are placed into a fund at the end of each year for the next 25 years. Five years after the last deposit, annual payments commence and continue forever. If $i = .09$, find the amount of each payment.

3-42. A loan of 5000 is repaid by annual payments continuing forever, the first one due one year after the loan is taken out. If the payments are $X, 2X, X, 2X, \ldots$ and $i = .16$, find X.

3-43. At what effective rate of interest is the present value of a series of payments of 1 at the end of every two years, forever, equal to 10?

3-44. Albert Glover has just signed a contract with the Blue Jays which will pay him 3,000,000 at the beginning of each year for the next five years. To finance his retirement, the player decides to put a part of each year's salary (the same amount each year) into a fund