MATHEMATICS OF INVESTMENT AND CREDIT

Fourth Edition

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ACTEX Publications, Inc.
Winsted, Connecticut

PREFACE

While teaching an intermediate level university course in mathematics of investment over a number of years, I found an increasing need for a textbook that provided a thorough and modern treatment of the subject, while incorporating theory and applications. This book is an attempt (as a 4^{th} edition, it must be a fourth attempt) to satisfy that need. It is based, to a large extent, on notes that I developed while teaching and my use of a number of textbooks for the course. The university course for which this book was written has also been intended to help students prepare for the mathematics of investment topic that is covered on one of the professional examinations of the Society of Actuaries and the Casualty Actuarial Society. A number of the examples and exercises in this book are taken from questions on past SOA/CAS examinations.

As in many areas of mathematics, the subject of mathematics of investment has aspects that do not become outdated over time, but rather become the foundation upon which new developments are based. The traditional topics of compound interest and dated cashflow valuations, and their applications, are developed in the first five chapters of the book. In addition, in Chapters 6 to 9, a number of topics are introduced which have become of increasing importance in modern financial mathematics over the past number of years. The past decade or so has seen a great increase in the use of derivative securities, particularly financial options. The subjects covered in Chapters 6 and 8 such as the term structure of interest rates and forward contracts form the foundation for the mathematical models used to describe and value derivative securities, which are introduced in Chapter 9.

The purpose of the methods developed in this book is to do financial valuations. This book emphasizes a direct calculation approach, assuming that the reader has access to a financial calculator with standard financial function.

The mathematical background required for the book is a course in calculus at the Freshman level. Chapter 9 introduces a couple of topics that involve the notion of probability, but mostly at an elementary level.

A very basic understanding of probability concepts should be sufficient background for those topics.

The topics in the first five Chapters of this book are arranged in an order that is similar to traditional approaches to the subject, with Chapter 1 introducing the various measures of interest rates, Chapter 2 developing methods for valuing a series of payments, Chapter 3 considering amortization of loans, Chapter 4 covering bond valuation, and Chapter 5 introducing the various methods of measuring the rate of return earned by an investment.

The content of this book is probably more than can reasonably be covered in a one-semester course at an introductory or even intermediate level. At the University of Toronto, the course on this subject is taught in two consecutive one-semester courses at the Sophomore level.

I would like to acknowledge the support of the Actuarial Education and Research Foundation, which provided support for the early stages of development of this book. I would also like to thank those who provided so much help and insight in the earlier editions of this book: John Mereu, Michael Gabon, Steve Linney, Walter Lowrie, Srinivasa Ramanujam, Peter Ryall, David Promislow, Robert Marcus, Sandi Lynn Scherer, Marlene Lundbeck, Richard London, David Scollnick and Robert Alps

I have had the benefit of many insightful comments and suggestions for this edition of the book from Keith Sharp, Louis Florence, Rob Brown, and Matt Hassett. I want to give a special mention of my sincere appreciation to Warren Luckner of the University of Nebraska, whose extremely careful reading of both the text and exercises caught a number of errors in the early drafts of this edition.

Marilyn Baleshiski is the format and layout editor, and Gail Hall is the mathematics editor at ACTEX. It has been a great pleasure for me to have worked with them on the book.

Finally, I am grateful to have had the continuous support of my wife, Sue Foster, throughout the development of each edition of this book.

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September 2008

incentive is a retractable-extendible feature, which gives the *bondholder* the option of having the bond redeemed (retracted) on a specified date, or having the redemption date extended to a specified later date. This is similar to a callable bond with the option in the hands of the bondholder rather than the bond issuer. Another incentive is to provide warrants with the bond. A warrant gives the bondholder the option to purchase additional amounts of the bond at a later date at a guaranteed price.

4.3.2 SERIAL BONDS AND MAKEHAM'S FORMULA

A bond issue may consist of a collection of bonds with a variety of redemption dates, or redemption in installments. This might be done so that the bond issuer can stagger the redemption payments instead of having a single redemption date with one large redemption amount. Such an issue can be treated as a series of separate bonds, each with its own redemption date, and it is possible that the coupon rate differs for the various redemption dates. It may also be the case that purchasers will want different yield rates for the different maturity dates. This bond is called a ***serial bond*** since redemption occurs with a series of redemption payments.

Suppose that a serial bond has redemption amounts $F_1, F_2, \ldots, F_m$, to be redeemed in $n_1, n_2, \ldots, n_m$ coupon periods, respectively, and pays coupons at rates $r_1, r_2, \ldots, r_m$, respectively, Suppose also that this serial bond is purchased to yield $j_1, j_2, \ldots, j_m$, respectively, on the m pieces. Then the price of the t^{th} piece can be formulated using any one of Equation (4.2), (4.3) or (4.4). Using Makeham's bond price formula given by Equation (4.4), the price of the t^{th} piece is

$$P_t \ = \ K_t + \frac{r_t}{j_t}(F_t - K_t), \qquad (4.12)$$

where $K_t = F_t \cdot v_{j_t}^{n_t}$. The price of the total serial issue would be $P = \sum_{t=1}^{m} P_t.$

In the special case where the coupon rates and yield rates on all pieces of the serial issue are the same, the total price of the issue can be written in a compact form using Makeham's Formula:

$$P \ = \ \sum_{t=1}^{m} P_t \ = \ \sum_{t=1}^{m}\left[K_t + \frac{r}{j}(F_t - K_t)\right] \ = \ K + \frac{r}{j}(F - K), \qquad (4.13)$$

where $K = \sum_{t=1}^{m} K_t$ is the present value of all redemption amounts for the entire issue, and $F = \sum_{t=1}^{m} F_t$ is the total redemption amount for the issue.

If the series of redemptions has a systematic form, such as a level amount every period for a number of periods, then K can be conveniently formulated as the present value of the annuity formed by the series of redemption amounts. Note that Equation (4.13) requires a uniform coupon rate and yield rate for all redemption dates in the issue.

EXAMPLE 4.7 (*Serial bond*)

On August 15, 2000 a corporation issues a 10% serial bond with face amount 50,000,000. The redemption is scheduled to take place at 5,000,000 every August 15 from 2010 to 2014 and 25,000,000 on August 15, 2015. Find the price of the entire issue on the issue date at a yield of $i^{(2)} = .125$.

SOLUTION

The present value of all of the redemption payments is

$$\begin{aligned} K &= 5{,}000{,}000\left[v_{.0625}^{20} + v_{.0625}^{22} + v_{.0625}^{24} + v_{.0625}^{26} + v_{.0625}^{28}\right] \\ &\quad + 25{,}000{,}000 \cdot v_{.0625}^{30} = 9{,}976{,}960. \end{aligned}$$

Then the price of the serial bond is

$$\begin{aligned} P &= K + \frac{r}{j}(F-K) \\ &= 9{,}976{,}960 + \frac{.05}{.0625}(50{,}000{,}000 - 9{,}976{,}960) = 41{,}995{,}392. \quad \square \end{aligned}$$

4.4 DEFINITIONS AND FORMULAS

Definition 4.1 – Bond

A bond is an interest-bearing certificate of public (government) or private (corporate) indebtedness.

Bond notation

- F – The face amount (also called the par value) of the bond
- r – the coupon rate per coupon period (six months unless otherwise specified)
- C – the redemption amount on the bond (equal to F unless otherwise noted)
- n – the number of coupon periods until maturity
- j – the yield rate per coupon period

Bond price on a coupon date

$$P = C\frac{1}{(1+j)^n} + Fr\left[\frac{1}{1+j} + \frac{1}{(1+j)^2} + \cdots + \frac{1}{(1+j)^n}\right]$$

$$= Cv_j^n + Fr\, a_{\overline{n|}\,j} \qquad (4.1)$$

$$= C + (Fr - Cj)a_{\overline{n|}\,j} \qquad (4.3)$$

Definition 4.2 – Bond Purchase Value

(a) If $P > F$, the bond is said to be bought **at a premium**.
(b) If $P = F$, the bond is said to be bought **at par**.
(c) If $P < F$, the bond is said to be bought **at a discount**.

Bond price between coupon dates

$P_t = P_0(1+j)^t$ is the price including accrued coupon at fraction t into the coupon period, where P_0 is the price just after the last coupon. The market price is $P_0(1+j)^t - Frt$ (the purchase price minus accrued coupon). (4.8)

$$t = \frac{\textit{number of days since last coupon paid}}{\textit{number of days in the coupon period}}. \qquad (4.7)$$

Amortization of a Bond

BV denotes the book value (or amortized value) and j is the yield rate.

$$BV_{t+1} = BV_t(1+j) - Fr \qquad (4.9)$$

$$I_{t+1} = BV_t \times j \qquad (4.10)$$

$$PR_{t+1} = Fr - I_{t+1} \qquad (4.11)$$

Definition 4. 3 – Callable Bond

A callable bond is one for which there is a range of possible redemption dates. The redemption is chosen by the bond issuer.

4.5 NOTES AND REFERENCES

The reference book *Standard Securities Calculation Methods*, provides a comprehensive collection of calculation and quotation methods used in financial practice. *The Handbook of Fixed Income Securities*, by F. Fabozzi covers a wide range of topics on bonds and other fixed income investments. The text by Butcher and Nesbitt details several numerical procedures for approximating yield rates, and provides additional references on the subject.

4. 6 EXERCISES

Unless specified otherwise, it is assumed that all coupon rates are quoted as annual rates but payable semiannually, all yield rates are nominal annual rates convertible semiannually, and bonds are valued just after a coupon has been paid.

SECTION 4.1

4.1.1 Find the prices of the following bonds, all redeemable at par. Show how to compare their prices without actually calculating the numerical values.

(a) A 10-year 100, 5% bond yielding 7.2%
(b) A 10-year 100, 5½ % bond yielding 7.7%
(c) A 12-year 100, 5% bond yielding 7.2%
(d) A 12-year 100, 5½ % bond yielding 7.7%

4.1.2 A twelve-year 100 par value bond pays 7% coupons semiannually. The bond is priced at 115.84 to yield an annual nominal rate of 6% compounded semiannually. Calculate the redemption value of the bond.

4.1.3 A zero-coupon bond pays no coupons and only pays a redemption amount at the time the bond matures. Greta can buy a zero-coupon bond that will pay 10,000 at the end of 10 years and is currently selling for 5,083.49. Instead she purchases a 10% bond with coupons payable semi-annually that will pay 10,000 at the end of 10 years. If she pays X she will earn the same annual effective interest rate as the zero coupon bond. Calculate X.

4.1.4 A 6% bond maturing in 8 years with semiannual coupons to yield 5% convertible semiannually is to be replaced by a 5.5% bond yielding the same return. In how many years should the new bond mature? (Both bonds have the same price, yield rate and face amount).

4.1.5 Don purchases a 1000 par value 10-year bond with 8% semiannual coupons for 900. He is able to reinvest his coupon payments at a nominal rate of 6% convertible semiannually. Calculate his nominal annual yield rate convertible semiannually over the ten-year period.

4.1.6 A 25-year bond with a par value of 1000 and 10% coupons payable quarterly is selling at 800. Calculate the annual nominal yield rate convertible quarterly.

4.1.7 An investor borrows an amount at an annual effective interest rate of 7% and will repay all interest and principal in a lump sum at the end of 10 years. She uses the amount borrowed to purchase a 1000 par value 10-year bond with 10% semiannual coupons bought to yield 8% convertible semiannually. All coupon payments are reinvested at a nominal rate of 6% convertible semiannually. Calculate the net gain to the investor at the end of 10 years after the loan is repaid.

4.1.8 In the table in Section 4.1 excerpted from the U.S. Bureau of Public Debt, a 5-year treasury bond is listed as having been issued on January 31, 2007 and maturing on January 31, 2012. The coupon rate is 4.75%, the yield rate at issue is listed as 4.855%, and the price at issue is listed as 99.539. Verify that this is the correct price for this bond.

4.1.9 The *National Post* ©, a Canadian daily newspaper, has listings after each trading day of the closing prices and yields of a number of bonds that traded the previous day. In the March 2, 2004 edition, there was the following listing for a Government of Canada bond:

Coupon	Maturity Date	Bid $	Yield %
4.25	Sep 01/09	102.76	3.69

(a) Verify that this is the correct price for the bond.

(b) In this listing, the price per $100 is rounded to the nearest $.01, and the yield rate is rounded to the nearest .01%. The quoted yield rate could be any number from 3.685% to 3.695% (and would be rounded to 3.69%, we can think of 3.695% as $3.694999\dot{9}\%$). Find the resulting prices at the two ends of that range of yield rates.

(c) The bid price of 102.76 could have been rounded from an actual price between 102.755 and 102.765. Find the yield rates that correspond to those prices.

4.1.10 An n-year 4.75% bond is selling for 95.59. An n-year 6.25% bond at the same yield would sell for 108.82. The face and redemption amount of the bond is 100. Find the yield rate.

4.1.11 Bond A has n coupons remaining at rate r_1 each, and sells to yield rate i_1 effective per coupon period. Bond B has the same face value and number of coupons remaining as Bond A, but the coupons are at rate r_2 each and the yield rate is i_2 effective per period. If $i_2 \cdot r_1 = i_1 \cdot r_2$ and $i_2 > i_1 > r_1$, which of the following statements are true?

I. The price of Bond B exceeds the price of Bond A.

II. The present value of Bond B's coupons on the purchase date exceeds the present value of Bond A's coupons.

III. The present value of the redemption amount for Bond B exceeds the corresponding present value for Bond A.

4.1.12 Two bonds, each of face amount 100, are offered for sale at a combined price of 240. Both bonds have the same term to maturity but the coupon rate for one is twice that of the other. The difference in price of the two bonds is 24. Prices are based on a nominal annual yield rate of 3%. Find the coupon rates of the two bonds.

4.1.13 A 7% bond has a price of 79.30 and a 9% bond has a price of 93.10, both per 100 of face amount. Both are redeemable in n years and have the same yield rate. Find n.

4.1.14 When a certain type of bond matures, the bondholder is subject to a tax of 25% on the amount of discount at which he bought the bond. A 1000 bond of this type has 4% *annually* paid coupons and is redeemable at par in 10 years. No tax is paid on coupons. What price should a purchaser pay to realize an effective annual yield of 5% after taxes?

4.1.15 Smith purchases a 20-year, 8%, 1000 bond with semiannual coupons. The purchase price will give a nominal annual yield to maturity, of 10%. After the 20^{th} coupon, Smith sells the bond. At what price did he sell the bond if his actual nominal annual yield is 10%?

4.1.16 Show that Equations (4.6a) and (4.6b) are algebraically equivalent.

4.1.17 In the bond quotations of a financial newspaper, a quote was given for the price on February 20, 2004 of an 11% bond with face amount 100 maturing on April 1, 2023. The yield was quoted as 11.267%. Find the quoted price to the nearest .001.

4.1.18 Suppose the redemption amount C is not necessarily equal to the face amount F on a bond. Using $g = \frac{Fr}{C}$ as the *modified coupon rate*, show that Equations (4.2), (4.3) and (4.4) become

$$P = C \cdot v_j^n + Cg \cdot a_{\overline{n}|j}, \tag{4.2E}$$

$$P = C + C(g-j) \cdot a_{\overline{n}|j}, \tag{4.3E}$$

and

$$P = K + \frac{g}{j}(C-K). \tag{4.4E}$$

Describe the relationship linking the relative sizes of P and C to the relative sizes of g and j.

4.1.19 A 1000 bond bearing coupons at annual rate 6.5%, payable semiannually, and redeemable at 1050 is bought to yield a nominal rate of 8%. If the present value of the redemption amount is 210, what is the price to the nearest 10?

4.1.20 A company issues 1,000,000 in bonds. The prevailing yield rate on the bonds is 12%. The company considers having coupons at 8% and a maturity of 15 years. On second thought, the company decides on a maturity date of 20 years. What coupon rate must the bond issue have in order for the company to raise the same amount of revenue as it would have on the 15-year issue? Suppose the company issued the bonds with a maturity date of 10 years. What coupon rate is required to raise the same amount as under the other two issues?

4.1.21S You have decided to invest in two bonds. Bond X is an n-year bond with semi-annual coupons, while bond Y is zero-coupon bond, which is redeemable in $n/2$ years. The desired yield rate is the same for both bonds. You also have the following information:

Bond *X*

- Par value is 1000.
- The ratio of the semi-annual bond rate to the desired semi-annual yield rate, $\frac{r}{i}$ is 1.03125.
- The present value of the redemption value is 381.50.

Bond *Y*

- Redemption value is the same as that of bond X.
- Price to yield is 647.80.

What is the price of bond X?

*4.1.22 Consider two bonds, each with face amount 1. One bond matures 6 months from now and carries one coupon of amount r_1. The other bond matures 1 year from now and carries two semiannual coupons of amount r_2 each. Both bonds have the same selling price to yield nominal annual $i^{(2)}$. Find a formulation for $i^{(2)}$ in terms of r_1, r_2 and constants.

*4.1.23 A bond issue carries quarterly coupons of 2% of the face amount outstanding. An investor uses Makeham's Formula to evaluate the whole outstanding issue to yield an effective annual rate of 13%. Find the value of H used in the formula $P = K + H(C - K)$.

*4.1.24 A bond with face and redemption amount of 3000 with *annual* coupons is selling at an effective annual yield rate equal to twice the annual coupon rate. The present value of the coupons is equal to the present value of the redemption amount. What is the selling price?

*4.1.25 On November 1, 1999 Smith paid 1000 for a government savings bond of face amount 1000 with annual coupons of 8%, with maturity to occur on November 1, 2011. On November 1, 2005 the government issues new savings bonds with the same maturity date of November 1, 2011, but with annual coupons of 9.5% (Smith's bond will still pay 8%). The government offers Smith a cash bonus of X to be paid on the maturity date if he holds his old bond until maturity. Smith can cash in his old bond on November 1, 2005 and buy a new bond for 1000. If both options yield 9.5% from November 1, 2005 to November 1, 2011, find X.

*4.1.26 Show that if $(1+j)^t$ is approximated by $1+jt$, then the quoted price of a bond at time t, $0 \leq t \leq 1$, since the last coupon is the linearly-interpolated value at t between P_0 and P_1. (This is the linearly interpolated price exclusive of the accrued coupon.)

*4.1.27 Show that $P_0(1+j) - Fr = P_1$. Then assuming that $r = j$ and $P_0 = F$, show that $P_1 = F$.

*4.1.28 Suppose that a bond has semiannual coupons of amount Fr each. At six-month effective yield rate j, a continuous payment for six months equivalent to a semiannual coupon is $\bar{r} \cdot F = (Fr / \bar{s}_{\overline{1}|j})$.

Suppose that the quoted price at time t (where $0 \leq t \leq 1$ is measured since the last coupon) is redefined to be the price-plus-accrued minus $\bar{r} \cdot F \cdot \bar{s}_{\overline{t}|j}$. Show that the quoted price in part (ii) of Example 4.2 would then become exactly 100 per 100 of face amount. Show that $\bar{r} \cdot \bar{s}_{\overline{t}|j}$ is approximately equal to $t \cdot Fr$.

*4.1.29 During the time when compound interest calculations were done by hand and with reference to interest tables, bond tables were constructed listing prices at issue (per 100 of face amount) of bonds with varying maturity dates, coupon rates and yield rates. Thus the bond price $P(n,r,j)$ is written as a function of *n*, *r* and *j*. Show that $P(n,r,j)$ is a linear function of *r* but not of *n* or *j*. Thus linear interpolation with respect to the coupon rate gives exact results, but linear interpolation with respect to the yield rate gives approximate results.

*4.1.30 Let $n \geq 1$, $0 \leq t \leq 1$, and $g(j) = [F \cdot v_j^n + Fr \cdot a_{\overline{n}|j}](1+j)^t$.

(a) Show that $g(j)$ is strictly decreasing and convex (i.e.,

$$g'(j) < 0 \text{ and } g''(j) > 0).$$

(b) Show that $\lim_{j \to -0} g(j) = +\infty$ and $\lim_{j \to \infty} g(j) = 0$.

(c) Use parts (a) and (b) to show that if $P > 0$ the equation

$$P = [F \cdot v_j^n + Fr \cdot a_{\overline{n}|j}](1+j)^t$$

has a unique solution for *j*.

SECTION 4.2

4.2.1 Find the total amount paid, the total interest and the total principal repaid in the amortization of Table 4.2.

4.2.2 Graph OB_k for each of the three cases in Example 4.4.

(a) Show that for a bond bought at a premium, the graph of OB_k is concave downward.

(b) Show that for a bond bought at a discount, the graph of OB_k is convex upward.