CHAPTER 4
DISCRETE FREQUENCY-SEVERITY INSURANCE MODEL UNDER INDEPENDENCE

This chapter presents a simple two-stage model of an insurance operation, based on Examples 2.2 and 2.3. Specifically, we assume that there is a single insured whose claim experience is modeled in the following manner. First, a die and spinner are selected independently and at random. Then using the notation of Chapter 2,

$$\begin{aligned} P\left[A_i \text{ and } B_j\right] &= P\left[A_i\right] \cdot P\left[B_j\right] \\ &= \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4}, \end{aligned} \tag{4.1}$$

for $i = 1, 2$ and $j = 1, 2$. (Once selected, the die and spinner, which determine the risk characteristics of the insured, are not replaced.) The random claims process begins when the selected die is rolled. If a marked face appears, a claim has occurred; if not, there is no claim. Then if there is a claim, the selected spinner is spun to determine the amount of the claim. Each roll of the die and spin of the spinner, if necessary, constitutes a single period of observation. We let X_i denote the random variable representing the (aggregate) amount of claims during the i^{th} period of observation, $i = 1, 2, \ldots$.

In this chapter, we first compute the initial **pure** (net) **premium**, $E[X_1]$, using the initial (*i.e.*, prior) probabilities $P[A_i \text{ and } B_j]$. Then, having observed the result of the first period of observation, we compute a revised pure premium estimate, $E[X_2 \mid X_1]$, based on the revised (*i.e.*, posterior) probabilities $P[A_i \text{ and } B_j \mid X_1]$ given the result of the first period of observation.

4.1 INITIAL PURE PREMIUM ESTIMATES

Since X_1 can take only the values 0, 2, and 14 with positive probability, we may write $E[X_1]$ as

$$E[X_1] = 0 \cdot P[X_1 = 0] + 2 \cdot P[X_1 = 2] + 14 \cdot P[X_1 = 14]. \quad (4.2)$$

Recalling the definitions of U_1 and M_1 from Example 2.2, we find

$$\begin{aligned} P[X_1 = 0] &= P[U_1] \\ &= P[U_1 \mid A_1] \cdot P[A_1] + P[U_1 \mid A_2] \cdot P[A_2] \\ &= \left(\frac{5}{6}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{2}{3}, \end{aligned} \quad (4.3)$$

and

$$\begin{aligned} P[X_1 = 2] &= P[M_1 \text{ and } (S_1 = 2)] \\ &= P[M_1] \cdot P[S_1 = 2] \\ &= (P[M_1 \mid A_1] \cdot P[A_1] + P[M_1 \mid A_2] \cdot P[A_2]) \\ &\quad \times (P[S_1 = 2 \mid B_1] \cdot P[B_1] + P[S_1 = 2 \mid B_2] \cdot P[B_2]) \\ &= \left[\left(\frac{1}{6}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\right] \times \left[\left(\frac{5}{6}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\right] \\ &= \frac{2}{9}. \end{aligned} \quad (4.4)$$

Finally,

$$\begin{aligned} P[X_1 = 14] &= 1 - P[X_1 = 0] - P[X_1 = 2] \\ &= 1 - \frac{2}{3} - \frac{2}{9} = \frac{1}{9}. \end{aligned} \quad (4.5)$$

The initial (prior) probabilities obtained above are summarized in column (2) of Table 4.1, shown on page 41.

TABLE 4.1

INITIAL PURE PREMIUM ESTIMATE		
(1) Value of x	(2) Initial Probability $P[X_1 = x]$	(3) $x \cdot P[X_1 = x]$ (1) × (2)
0	6/9	0
2	2/9	4/9
14	1/9	14/9
Totals	1	2

By adding the entries in column (3) of Table 4.1, we find the initial estimate of the pure premium to be $E[X_1] = 2$.

We can also write $E[X_1]$ as

$$E[X_1] = \sum_{i=1}^{2} \sum_{j=1}^{2} E[X_1 \mid A_i \text{ and } B_j] \cdot P[A_i \text{ and } B_j]\,, \tag{4.6}$$

where $E[X_1 \mid A_i \text{ and } B_j]$ is the mean claim amount, given the pair of events A_i and B_j. It, in turn, is equal to the product of (a) the mean number of claims, and (b) the mean severity amount, given that a claim occurs. Symbolically this is expressed as

$$E[X_1 \mid A_i \text{ and } B_j] = E[I_1 \mid A_i] \cdot E[S_1 \mid B_j],$$

where

$$I_1 = \begin{cases} 1 & \text{if the first toss of the die produces a marked side} \\ 0 & \text{if otherwise} \end{cases}.$$

Thus it follows that $E[I_1 \mid A_i] = P[M_1 \mid A_i]$. From Examples 2.2 and 2.3, we obtain the results shown in Table 4.2 on page 42. Substituting the pure premium values from column (4), along with the probabilities $P[A_i \text{ and } B_j] = \frac{1}{4}$, into Equation (4.6) above, we again obtain the result $E[X_1] = 2$.

TABLE 4.2

PURE PREMIUM BY TYPE OF DIE AND SPINNER			
(1) Type of Die and Spinner	(2) Frequency $E[I_1 \mid A_i]$	(3) Severity $E[S_1 \mid B_j]$	(4) $E[X_1 \mid A_i \text{ and } B_j]$ (2) × (3)
A_1 and B_1	1/6	4	2/3
A_1 and B_2	1/6	8	4/3
A_2 and B_1	1/2	4	2
A_2 and B_2	1/2	8	4

4.2 REVISED PURE PREMIUM ESTIMATES AND PREDICTIVE DISTRIBUTIONS

In this section we estimate the pure premium for the second period of observation, given the result of the first period. Symbolically we seek the values of $E[X_2 \mid X_1 = k]$, for $k = 0, 2,$ and 14. The method of solution also yields the conditional probabilities of X_2, given the value of X_1. These conditional probabilities constitute the **predictive distribution of the random variable X_2, given the value of the random variable X_1**.

Since (a) once a die has been chosen, the result of each toss of that die is independent of the results of all other tosses, and (b) once a spinner has been selected, the result of each spin of that spinner is independent of the results of all other spins, then the random variable X_2 depends only on the result of the first period of observation through the joint probabilities of A_i and B_j. In other words, the only effect is through the *revised* die-spinner probabilities $P[A_i \text{ and } B_j \mid X_1 = k]$.

By analogy with Equation (4.2), we may write

$$\begin{aligned} E[X_2 \mid X_1 = k] & \\ &= 0 \cdot P[X_2 = 0 \mid X_1 = k] + 2 \cdot P[X_2 = 2 \mid X_1 = k] \\ &\quad + 14 \cdot P[X_2 = 14 \mid X_1 = k], \end{aligned} \tag{4.7}$$

for $k = 0$, 2, and 14. Our goal is to calculate the conditional expected claim amount (pure premium) for the second period of observation, after having observed a claim amount of k during the first period of observation. For the reasons given in the preceding section, we may write

$$P[X_2 = m \mid X_1 = k] = \sum_{i=1}^{2} \sum_{j=1}^{2} P[X_2 = m \mid A_i \text{ and } B_j] \times P[A_i \text{ and } B_j \mid X_1 = k], \tag{4.8}$$

for $m = 0$, 2, and 14. The reader should note that the result

$$P[X_2 = m \mid A_i \text{ and } B_j] = P[X_1 = m \mid A_i \text{ and } B_j]$$

follows from the conditions of the model.

We first calculate the probabilities $P[X_1 = m \mid A_i \text{ and } B_j]$, starting with $m = 0$. In this case we have

$$P[X_1 = 0 \mid A_i \text{ and } B_j] = P[X_1 = 0 \mid A_i] = P[U_1 \mid A_i]\,. \tag{4.9}$$

The values of $P[U_1 \mid A_1] = \frac{5}{6} = \frac{30}{36}$ and $P[U_1 \mid A_2] = \frac{1}{2} = \frac{18}{36}$ are already known. Next, we consider $P[X_1 = 2 \mid A_i \text{ and } B_j]$. We know that

$$P[X_1 = 2 \mid A_i \text{ and } B_j] = P[M_1 \mid A_i] \cdot P[S_1 = 2 \mid B_j], \tag{4.10}$$

so these values can be easily calculated, as shown in Table 4.3.

TABLE 4.3

$P[X_1 = 2 \mid A_i$ and $B_j]$

(1) Type of Die and Spinner	(2) $P[M_1 \mid A_i]$	(3) $P[S_1 = 2 \mid B_j]$	(4) $P[X_1 = 2 \mid A_i$ and $B_j]$ (2) × (3)
A_1 and B_1	1/6	5/6	5/36
A_1 and B_2	1/6	1/2	3/36
A_2 and B_1	1/2	5/6	15/36
A_2 and B_2	1/2	1/2	9/36

The calculation of $P[X_1 = 14 \mid A_i \text{ and } B_j]$ is left to the reader as Exercise 4-2. The results of Equation (4.9), Equation (4.10), and Exercise 4-2 are summarized in Table 4.4. Note that the $m = 2$ column comes from column (4) of Table 4.3.

TABLE 4.4

PROBABILITY OF CLAIM OUTCOME GIVEN DIE AND SPINNER
$\mathbf{P[X_1 = m \mid A_i \text{ and } B_j]}$

Type of Die and Spinner	Value of m: 0	2	14	Total
A_1 and B_1	30/36	5/36	1/36	1
A_1 and B_2	30/36	3/36	3/36	1
A_2 and B_1	18/36	15/36	3/36	1
A_2 and B_2	18/36	9/36	9/36	1

In order to evaluate the right side of Equation (4.8), we need the values of $P[A_i \text{ and } B_j \mid X_1 = k]$, for $k = 0, 2,$ and 14. From Bayes' theorem we have

$$P[A_i \text{ and } B_j \mid X_1 = k] = \frac{P[X_1 = k \mid A_i \text{ and } B_j] \cdot P[A_i \text{ and } B_j]}{P[X_1 = k]}. \quad (4.11)$$

Since (a) $P[A_i \text{ and } B_j] = \frac{1}{4}$ for all i and j, (b) the values of the initial probabilities $P[X_1 = k]$ are given in column (2) of Table 4.1, and (c) the values of $P[X_1 = k \mid A_i \text{ and } B_j]$ are given in Table 4.4, we can easily evaluate $P[A_i \text{ and } B_j \mid X_1 = k]$, for $k = 0$, 2, and 14. For example, Equation (4.11) produces

$$P[A_1 \text{ and } B_2 \mid X_1 = 14] = \frac{P[X_1 = 14 \mid A_1 \text{ and } B_2] \cdot P[A_1 \text{ and } B_2]}{P[X_1 = 14]} = \frac{\left(\frac{3}{36}\right)\left(\frac{1}{4}\right)}{\frac{1}{9}} = \frac{3}{16}.$$

All of the results from Equation (4.11) are summarized in the following table.

TABLE 4.5

POSTERIOR DISTRIBUTION OF DIE-SPINNER COMBINATIONS $P[A_i \text{ and } B_j \mid X_1 = k]$			
Type of Die and Spinner	Value of k 0	2	14
A_1 and B_1	5/16	5/32	1/16
A_1 and B_2	5/16	3/32	3/16
A_2 and B_1	3/16	15/32	3/16
A_2 and B_2	3/16	9/32	9/16
Totals	1	1	1

Tables 4.4 and 4.5 give the results needed to evaluate the conditional probabilities of Equation (4.8). For example

$$
\begin{aligned}
&P[X_2 = 2 \mid X_1 = 14] \\
&\quad = \sum_{i=1}^{2} \sum_{j=1}^{2} P[X_2 = 2 \mid A_i \text{ and } B_j] \cdot P[A_i \text{ and } B_j \mid X_1 = 14] \\
&\quad = \left(\frac{5}{36}\right)\left(\frac{1}{16}\right) + \left(\frac{3}{36}\right)\left(\frac{3}{16}\right) \\
&\qquad\qquad + \left(\frac{15}{36}\right)\left(\frac{3}{16}\right) + \left(\frac{9}{36}\right)\left(\frac{9}{16}\right) \\
&\quad = \frac{35}{144}.
\end{aligned}
$$

Table 4.6 summarizes the values of the conditional probabilities of X_2 given X_1. These values constitute the predictive distribution of the random variable X_2, given the value of X_1. That is, given the result of the first period of observation, the appropriate column of Table 4.6 gives the probability of each possible outcome for the second period.

TABLE 4.6

PREDICTIVE DISTRIBUTION OF X_2 GIVEN X_1
$P[X_2 = m \mid X_1 = k]$

m	Value of k: 0	2	14
0	51/72	168/288	84/144
2	14/72	85/288	35/144
14	7/72	35/288	25/144
Totals	1	1	1

Finally, we can use Equation (4.7) and the entries of Table 4.6 to calculate the conditional expectations of X_2, given X_1. For example, for $k = 2$,

$$\begin{aligned} E[X_2 \mid X_1 = 2] &= 2 \cdot P[X_2 = 2 \mid X_1 = 2] \\ &\quad + 14 \cdot P[X_2 = 14 \mid X_1 = 2] \\ &= 2\left(\frac{85}{288}\right) + 14\left(\frac{35}{288}\right) = \frac{55}{24}. \end{aligned}$$

The results for all three values of k are given in the following table.

TABLE 4.7

k	$E[X_2 \mid X_1 = k]$
0	42/24
2	55/24
14	70/24

An alternative approach to find $E[X_2 \mid X_1 = k]$ is to directly calculate the joint probabilities $P[X_2 = m \text{ and } X_1 = k]$, and to then obtain the entries of Table 4.6 by using the definition of conditional probability

$$P[X_2 = m \mid X_1 = k] = \frac{P[X_2 = m \text{ and } X_1 = k]}{P[X_1 = k]}. \qquad (4.12)$$