PREFACE

The discipline of life contingencies has for many generations defined the basic principles of the work of the actuary to a far greater extent than any other single subject, technical or otherwise. Many actuarial students, their employers, and their professors have considered the moment of success on the Society of Actuaries examination covering life contingencies as a point of metamorphosis from the status of neophyte to that of a full-fledged prospect for membership and meaningful contributions in his or her chosen profession.

Every profession deserves a single reference which seeks to define its most fundamental principles, articulately and with technical precision. The actuarial profession is fortunate to have such a reference in *Actuarial Mathematics* (to be referred to herein as "the text") as an unquestioned authority for all who seek membership. It is an enjoyable but also challenging text from which to teach, as many enlightening digressions must be made in the classroom which expand students' thought processes beyond the mathematically-precise theory and illustrations.

Having worked in a face-to-face classroom environment with more than 2500 students who have passed the relevant Society of Actuaries or Casualty Actuarial Society Examination, one conclusion has become clear. Regardless of its quality, the student who studies only the prescribed text is at a serious competitive disadvantage. Its exercises do not bear a close resemblance to those constructed by the Examination Committees. Its derivations, though mathematically sound, are often of such length and technicality that their true significance is masked. Perhaps it could even be argued that its heavy emphasis on statistics clouds some of the relationships which are truly more logical than statistical.

This volume, notwithstanding its supplementary concepts, its formulas, its techniques, and its exercises and detailed solutions, should never be considered as a substitute for the text itself. This is strictly a supplement which is not self-contained and would be found to be of little value in the absence of an accompanying detailed study of *Actuarial Mathematics*.

Much of the material contained herein was developed while instructing students of life contingencies at Georgia State University. Upon completion of the course sequence, consistently over 90% of the students were successful on the exam on their first attempt. The exercises, with few exceptions, are original problems to which these young people were exposed throughout their course of study. They were not created for exam-passing purposes, but rather to provide the students a source of questions designed to test, and to expose their weaknesses on, the basic principles of life contingencies.

The format of this manual should assist the student from his or her first introduction to the material through the often frantic days of final preparation. Each chapter begins with supplementary concepts and examples designed to shed light on those topics which the reader of Bowers may not have assimilated sufficiently. At the end of each chapter are practice problems followed by solutions. These 360 problems should be used, a few at a time, to test knowledge on topics as they are traditionally examined by the actuarial societies.

Following the chapter summaries is a section consisting of solutions to 87 carefully chosen questions from the Bowers text. Before attempting the text exercises, it is recommended that the student carefully read the introduction to this section on Page 239 in this manual. Finally, four twentyquestion practice exams are included, preferably to be taken under simulated exam conditions in the final few weeks leading up to the exam date. Upon scoring the results and analyzing the given solutions for any careless errors or alternate techniques, the student should be prepared for whatever will be encountered on exam day.

Special thanks go to Gail Hall, FSA, MAAA, Vice President of Product Development at ACTEX, who originated the idea of developing a 2005 version of this manual and whose guidance throughout the process was invaluable. The typesetting was performed by Marilyn Baleshiski of ACTEX with great patience, good humor, and diligent attention to detailed notation and symbols unique to life contingencies. My gratitude goes to Professor Hal Pedersen, ASA, of the University of Manitoba, for his careful reading of the manuscript which created the opportunity for expanded explanations in addition to a final search for errors. I am also grateful to the students with whom I have worked over a 40 year period in their pursuit of mastery of the subject. Many of the ideas and techniques contained herein were refined by virtue of their incisive questions and comments.

My greatest level of appreciation goes to my wife, Jane, who performed much of the original typing on this manual. More importantly, she has been both a tireless supporter and a true inspiration to me throughout this and earlier projects.

Proofreading of material such as this can never be complete in spite of many efforts to ferret out every misplaced superscript or coefficient. The errors which remain are the full responsibility of the author; it is hoped, at least, that none of these errors are of a substantive nature.

Life contingencies examinations are passed far more readily by candidates who approach the topic logically and who employ insightful techniques than by the accomplished mathematician/statistician who treats it as a challenge to derive overcomplicating formulas under rigorous examination conditions. This volume represents an effort to convince examination candidates that this observation is accurate. Historically, many who have depended primarily on mathematical skills in their preparation have done so to their own detriment.

With the logical approaches incorporated in this manual, supplemented by intensive practice in problem-solving, the student of life contingencies should not only be successful on the first exam attempt but should also maximize his or her enjoyment of a subject which on the surface may seem depressing. Over the past forty years, even as my personal omega looms closer, I have increasingly enjoyed working with students of life contingencies. It is my wish that this enjoyment will be evident to the reader.

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Chapter Six

Supplementary Concepts

1. Whereas only two distinct "principles" are applied to premium determination in Chapter Six, there are many situations in which neither may be applied. The student must be keenly aware that by far the more important of these, the equivalence principle, may not be assumed unless at least one of the following is indicated:

   1) the statement that $E[L] = 0$, or simply that the equivalence principle (EP) is to be imposed.

   2) the use of the words "benefit premium,"

   3) the presence of a "punctuated" premium symbol, such as P_x, $\overline{P}(\overline{A}_x)$, ${}_nP_x$, or $\overline{P}({}_{n|}\overline{a}_x)$. Use of the generic "$P$" or just the word "premium" gives no indication of the principle, if any, which is being assumed.

   Note that setting up a premium equation of the form "actuarial present value of future premiums" equals "actuarial present value of future benefits" is inappropriate in the absence of an assumption of the equivalence principle.

2. The other named principle, although introduced in Chapter Six, is rarely used in practice and often produces major inconsistencies and absurdities. The "percentile premium principle" can be shown to produce premiums in some circumstances which are the same for term insurances as for endowment insurances, or even premiums equal to zero for certain term insurances.

   It should be clear that the following phrases are equivalent and, hence, interchangeable.

   1) "Find the smallest annual premium which an insurer may charge in order to incur a positive loss on no more than 15% of its policies."

   2) "Find the fifteenth percentile premium."

   Exercises 6-4, 6-5, and 6-27 illustrate questions of this type as well as the recommended approaches.

3. The variances of prospective loss random variables L are closely related to those of present value random variables for single premium insurance policies introduced in Chapter Four. The variance formulas are very compact where the equivalence principle is assumed, but are not as simple or as easy to remember in the absence of the equivalence principle.

Example 1

If $L = v^T - \overline{P}\overline{a}_{\overline{T}|}$, $T \geq 0$, then $VAR[L] = \dfrac{{}^2\overline{A}_x - (\overline{A}_x)^2}{(1-\overline{A}_x)^2}$ if the

equivalence principle is assumed, i.e., if $\overline{P} = \overline{P}(\overline{A}_x)$. Otherwise,

$$VAR[L] = [{}^2\overline{A}_x - (\overline{A}_x)^2]\left(1 + \frac{\overline{P}}{\delta}\right)^2.$$

Note that each of these variance formulas shows clearly that $VAR[L] > VAR[Z]$, where Z is the present value random variable for continuous whole life insurance in Chapter Four. This reflects the greater "uncertainty" inherent in L because the premiums are continuous for life rather than being paid in full at $t = 0$.

Example 2

If

$$\begin{aligned} L &= v^T - \overline{P}\overline{a}_{\overline{T}|}, \quad 0 \leq T \leq n \\ &= v^n - \overline{P}\overline{a}_{\overline{n}|}, \quad T > n, \end{aligned}$$

then

$$VAR[L] = \frac{{}^2\overline{A}_{x:\overline{n}|} - (\overline{A}_{x:\overline{n}|})^2}{(1-\overline{A}_{x:\overline{n}|})^2}$$

if the equivalence principle is assumed, i.e., if $\overline{P} = \overline{P}(\overline{A}_{x:\overline{n}|})$.
Otherwise,

$$VAR[L] = [{}^2\overline{A}_{x:\overline{n}|} - (\overline{A}_{x:\overline{n}|})^2]\left(1 + \frac{\overline{P}}{\delta}\right)^2.$$

Analogous formulas exist for fully discrete whole life and endowment insurance loss random variables.

4. Unfortunately, simple variance formulas similar to those just stated do not exist for loss random variables associated with products other than whole life or endowment insurance. When variances of such random variables are required, they are best obtained through an appeal to basic principles or through a technique to be introduced in Chapter Eight.

5. The pure endowment annual premium $P_{x:\overline{n}|}^{\;\;1}$ equals the reciprocal of the actuarial accumulated value $\ddot{s}_{x:\overline{n}|}$. This is because the share of the survivor who has deposited $P_{x:\overline{n}|}^{\;\;1}$ at the beginning of each year for n years is the contractual \$1 pure endowment, i.e.,

$$P_{x:\overline{n}|}^{\;\;1}\ddot{s}_{x:\overline{n}|} = 1\,.$$

6. Level benefit premiums such as ${}_nP_x$, $P_{x:\overline{n}|}^{1}$, and $P_{x:\overline{n}|}$ purchase identical insurance between ages x and $x+n$. The difference in the magnitude of the premiums is solely attributable to the investment feature of the contract.

Accordingly, comparisons of the policy values of survivors at age $x+n$ may be effected by analyzing future benefits; past benefits may be ignored because they are equivalent.

Examples

$$({}_nP_x - P_{x:\overline{n}|}^{1})\,\ddot{s}_{x:\overline{n}|} = A_{x+n} = \frac{{}_nP_x - P_{x:\overline{n}|}^{1}}{P_{x:\overline{n}|}^{\;\;1}}$$

$$(P_{x:\overline{n}|} - {}_nP_x)\,\ddot{s}_{x:\overline{n}|} = 1 - A_{x+n} = \frac{P_{x:\overline{n}|} - {}_nP_x}{P_{x:\overline{n}|}^{\;\;1}}$$

$$(P_{x:\overline{n}|} - P_{x:\overline{n}|}^{1})\,\ddot{s}_{x:\overline{n}|} = 1 = \frac{P_{x:\overline{n}|} - P_{x:\overline{n}|}^{1}}{P_{x:\overline{n}|}^{\;\;1}}$$

Because of the form of the right-hand sides of the above relationships, the approach to these types of problems is often referred to as the "P minus P over P" technique.

These concepts will be extended after the introduction of the notion of reserves in Chapter Seven.

7. The two most useful identities in life contingencies relate the most basic quantities in Chapters Four, Five, and Six to each other. While virtually every student will be comfortable with the relationship

$$\bar{A}_{x:\overline{n}|} = 1 - \delta\bar{a}_{x:\overline{n}|},$$

fewer are fluent with the relationship

$$\bar{P}(\bar{A}_{x:\overline{n}|}) = \frac{1}{\bar{a}_{x:\overline{n}|}} - \delta$$

and its variations.

Every entry in the following tables should be instantly recognizable to the well-prepared student. Note that deletion of the "$\overline{n}|$" in each formula simply produces the corresponding whole life relationship. Further, note that none of these expressions may be adapted to policies other than whole life and n-year endowment insurances, such as term or limited-pay whole life.

$A_{x:\overline{n}\|} =$		$1 - d\ddot{a}_{x:\overline{n}\|}$	$\dfrac{P_{x:\overline{n}\|}}{P_{x:\overline{n}\|} + d}$
$\ddot{a}_{x:\overline{n}\|} =$	$\dfrac{1 - A_{x:\overline{n}\|}}{d}$		$\dfrac{1}{P_{x:\overline{n}\|} + d}$
$P_{x::\overline{n}\|} =$	$\dfrac{dA_{x:\overline{n}\|}}{1 - A_{x:\overline{n}\|}}$	$\dfrac{1}{\ddot{a}_{x:\overline{n}\|}} - d$	

$\overline{A}_{x:\overline{n}\|} =$		$1 - \delta\overline{a}_{x:\overline{n}\|}$	$\dfrac{\overline{P}(\overline{A}_{x:\overline{n}\|})}{\overline{P}(\overline{A}_{x:\overline{n}\|}) + \delta}$
$\overline{a}_{x:\overline{n}\|} =$	$\dfrac{1 - \overline{A}_{x:\overline{n}\|}}{\delta}$		$\dfrac{1}{\overline{P}(\overline{A}_{x:\overline{n}\|}) + \delta}$
$\overline{P}(\overline{A}_{x:\overline{n}\|}) =$	$\dfrac{\delta\overline{A}_{x:\overline{n}\|}}{1 - \overline{A}_{x:\overline{n}\|}}$	$\dfrac{1}{\overline{a}_{x:\overline{n}\|}} - \delta$	

8. If $L = v^T - \overline{P}(\overline{A}_x)\overline{a}_{\overline{T}|}$, $T \geq 0$, and if the forces of mortality and interest are μ and δ, respectively, the probability that the insurer will realize a gain (i.e., a negative loss) on a random insured is given by

$$\left(\frac{\mu}{\mu + \delta}\right)^{\mu/\delta}.$$

Compare this statement to Supplementary Concepts #17 in Chapter Four and #10 in Chapter Five.

Special Mortality Laws

I. Constant Force of Mortality

$$P_x = vq_x = P^1_{x:\overline{n}|}$$
$$\overline{P}(\overline{A}_x) = \mu = \overline{P}(\overline{A}^1_{x:\overline{n}|})$$

For fully discrete whole life,

$$VAR[L] = p \cdot {}^2A_x = \frac{pq}{q + 2i + i^2}$$

if the equivalence principle is assumed.

For fully continuous whole life,

$$VAR[L] = {}^2\overline{A}_x = \frac{\mu}{\mu + 2\delta}$$

if the equivalence principle is assumed.

II. Uniform Distribution of Deaths

$$P(\overline{A}_x) = \frac{i}{\delta} P_x$$

$$P(\overline{A}^{\,1}_{x:\overline{n}|}) = \frac{i}{\delta} P^1_{x:\overline{n}|}$$

$$P(\overline{A}_{x:\overline{n}|}) = \frac{i}{\delta} P^1_{x:\overline{n}|} + P_{x:\overline{n}|}^{\ \ 1}$$

$$P^{(m)}_x = \frac{P_x}{\alpha(m) - \beta(m)(P_x + d)}$$

$$P^{(m)}_{x:\overline{n}|} = \frac{P_{x:\overline{n}|}}{\alpha(m) - \beta(m)(P^1_{x:\overline{n}|} + d)}$$

$${}_nP^{(m)}_x = \frac{{}_nP_x}{\alpha(m) - \beta(m)(P^1_{x:\overline{n}|} + d)}$$

$${}_hP^{(m)}(\overline{A}^{\,1}_{x:\overline{n}|}) = \frac{i}{\delta} \cdot {}_hP^{1\,(m)}_{x:\overline{n}|}$$

Exercises

6-1 Rank in increasing numerical size:

i) $P(\overline{A}_x)$
ii) $\overline{P}(\overline{A}_x)$
iii) $P^{(4)}(\overline{A}_x)$
iv) $P^{(2)}(\overline{A}_x)$

6-2 Let $\mu(x) = \frac{1}{90 - x}$, $0 \le x < 90$, and $i = .06$.

Find the annual benefit premium for a fully discrete 10-year endowment insurance issued to a woman aged 60.

6-3 A fully continuous \$1 whole life policy is issued to (x). If the premium is based upon the equivalence principle, $VAR\,[L] = .01$. If the annualized premium is π, $VAR\,[L] = .16$.

If $\overline{A}_x = .20$ and $\overline{a}_x = 16$, find π.

Solutions

6-1 **Key Approach:** Since each of the premiums involves a numerator of $\bar{A}_x$, we must analyze the denominators which are, respectively, $\ddot{a}_x$, $\bar{a}_x$, $\ddot{a}_x^{(4)}$ and $\ddot{a}_x^{(2)}$. While formulas for these annuities could be compared, it is more instructive for the student to consider their relative magnitudes logically.

First, $\ddot{a}_x$ makes the full annual payment at the beginning of each year; hence it is the largest, i.e., most expensive, of the annuity values.

Second, $\ddot{a}_x^{(2)}$ exceeds $\ddot{a}_x^{(4)}$, as it makes larger payments sooner in each respective year.

Finally, $\ddot{a}_x^{(4)}$ exceeds $\bar{a}_x$, which can be thought of as $\ddot{a}_x^{(\infty)}$, for the same reason that $\ddot{a}_x^{(2)}$ exceeds $\ddot{a}_x^{(4)}$.

Thus, since the premiums are inversely related to their corresponding annuities, we have

$$\underline{\underline{(i) < (iv) < (iii) < (ii)}}.$$

6-2 **Key Approach:** Since the mortality basis here is DML, and since our basic tools for handling insurances involving DML were developed in Chapter Four, we should use the relationship

$$P_{x:\overline{n}|} = \frac{dA_{x:\overline{n}|}}{1-A_{x:\overline{n}|}}.$$

$$\begin{aligned} A_{60:\overline{10}|} &= A^{\,1}_{60:\overline{10}|} + {}_{10}E_{60} \\ &= \frac{a_{\overline{10}|}}{30} + v^{10}\frac{20}{30} \\ &= .6176. \end{aligned}$$

Therefore $P_{60:\overline{10}|} = \frac{.6176d}{1-.6176} = \underline{\underline{.0914}}$

6-3 **Key Technique:** When two different premium rates are being analyzed for a loss random variable L, it is usually efficient to consider a ratio when comparing the two variances.

Key Formulas: $VAR[L] = (1+\frac{\pi}{\delta})^2({}^2\overline{A}_x - (\overline{A}_x)^2)$ (FCWL)

$\overline{P}(\overline{A}_x) = \frac{\overline{A}_x}{\overline{a}_x}$

Letting L^* be the loss random variable when the premium is π, we have

$$\frac{VAR[L^*]}{VAR[L]} = \frac{\left(1+\frac{\pi}{\delta}\right)^2}{\left(1+\frac{\overline{P}(\overline{A}_x)}{\delta}\right)^2} = \left(\frac{\pi+\delta}{\overline{P}(\overline{A}_x)+\delta}\right)^2 = 16.$$

$$\therefore \quad (\pi+\delta)^2\,(\overline{a}_x)^2 = 16$$

$$\therefore \quad (\pi+\delta)\,\overline{a}_x = 4$$

Since $\delta = \frac{1-\overline{A}_x}{\overline{a}_x} = .05$, we have

$$\pi = \frac{4}{16} - .05 = \underline{\underline{.20}}.$$

6-4 a) **Key Concept:** Since $l_0 = 100$ and $l_5 = 75$, and since the insurer must be a "winner" in at least 75% of the cases, it must earn a profit for all deaths occurring after age 5, i.e., on all policies for which death occurs in the sixth year or later.

$$\therefore \pi \ddot{s}_{\overline{6}|.04} > 10{,}000$$

$$\therefore \pi > 1449.634$$

b) **Key Concept:** Since $l_0 = 100$ and $l_5 = 75$, and since the insurer cannot tolerate a loss in as many as 25% of the cases, it cannot be a "loser" on those policies for which deaths occur in the fifth policy year. Thus it must at least break even on the policies which become death claims in the fifth year.

$$\therefore \pi \ddot{s}_{\overline{5}|.04} \geq 10{,}000$$

$$\therefore \pi \geq 1775.261$$

Notes: The student should understand clearly the fine, but significant, difference in the two parts of this question. Further, it should be clear why each answer must be rounded up, producing final values of $1449.64 and $1775.27, respectively.

Introductory Note

As is often the case with mathematical textbooks which lean toward the theoretical, *Actuarial Mathematics* (Bowers, et al) contains many end-of-chapter exercises which are of limited applicability in preparation for a problem-solving professional examination such as Society of Actuaries Exam M.

This does not mean that proofs and derivations cannot be of immense value to actuarial students in their attempt to master the theory and application of life contingencies. Every textbook exercise which a student completes, whether it be sophisticated or elementary, symbolic or numerical, hard or easy, surely adds to overall understanding even if perhaps in some cases only marginally.

Nevertheless, some exercises are obviously of more benefit than others. In this section the reader will find solutions to 87 carefully selected questions from the Bowers text which seem to have special significance to the actuarial student of life contingencies. While it is hoped that the student will attempt other text exercises as well, as a minimum these 87 questions and solutions should be clearly understood by any serious candidate for Exam M.

Several additional observations should be made. First, these exercises should not be attempted (nor the solutions read) until other sections of this manual have been carefully studied. This study should include mastery of special mortality laws (such as deMoivre's Law, constant force of mortality, and Modified DeMoivre's Law). Second, while traditional solutions to exercises such as these rely heavily upon calculus (especially integral calculus), the student should make a serious attempt to minimize, and perhaps nearly eliminate, calculus-based solutions. Historically, professional examinations in life contingencies require integral calculus in no more than one or two questions per exam, although many others could be solved using calculus but often in unnecessarily longer and less intuitive ways. You will notice that very few solutions contained herein involve integration and many emphasize logical, less sophisticated techniques. Third, the student should attempt each of these exercises in a serious manner before looking at the given solutions. If a first effort produces a correct answer, but is obtained in a relatively inefficient manner, only then will the student be able to gain maximum benefit from the solutions contained here.

Too many students of the Bowers text have ignored its exercises and, in many cases, even the text itself. Often this approach has been to the student's great detriment. While it may have worked for some, it has been an unwise decision for many others. The concepts illustrated by most of these 87 problems have appeared in similar form on the SOA Course 150 Exam and SOA/CAS Exam 3 over the last decade. The student who has not studied them or who approaches them with purely mathematical techniques has missed out on a gold mine of valuable material for exam preparation.

6.6 $$\frac{{}^2\bar{A}_x - (\bar{A}_x)^2}{(1-\bar{A}_x)^2} = \frac{\frac{\mu}{\mu+2\delta} - \left(\frac{\mu}{\mu+\delta}\right)^2}{\left(\frac{\delta}{\mu+\delta}\right)^2}$$

Somewhat tedious algebra produces the result $\frac{{}^2\bar{A}_x - (\bar{A}_x)^2}{(1-\bar{A}_x)^2} = {}^2\bar{A}_x = \frac{\mu}{\mu+2\delta}$

6.7 **Key Facts**:

$$\bar{P}(\bar{A}_x) = \frac{1}{\bar{a}_x} - \delta$$

$$\bar{a}_x = \mathring{e}_x \quad \text{if} \quad i = 0$$

$\therefore$ With $\delta = i = 0$,

$$\bar{P}(\bar{A}_x) = \frac{1}{\mathring{e}_x}$$

6.8 **Key Observations**: For continuous whole life (Chapter 4),

$$VAR[Z] = {}^2\bar{A}_x - (\bar{A}_x)^2.$$

For fully continuous whole life (Chapter 6),

$$VAR[L] = \frac{{}^2\bar{A}_x - (\bar{A}_x)^2}{(1-\bar{A}_x)^2}.$$

Since $\bar{A}_x$ is between 0 and 1, clearly $VAR[L] > VAR[Z]$, proving the assertion.

6.11 $${}_{20}P({}_{20|10}A_x) = \frac{{}_{20|10}A_x}{\ddot{a}_{x:\overline{20|}}} = \frac{A^1_{x:\overline{30|}} - A^1_{x:\overline{20|}}}{\ddot{a}_{x:\overline{20|}}}$$

$${}_{20}P^1_{x:\overline{30|}} - P^1_{x:\overline{20|}} = \frac{A^1_{x:\overline{30|}} - A^1_{x:\overline{20|}}}{\ddot{a}_{x:\overline{20|}}}$$

Clearly, the identity is shown to be correct.